Vacuum structure of two-dimensional ϕ^4 theory on the orbifold S^1/Z_2

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We consider the vacuum structure of two-dimensional ϕ^4 theory on S^1/Z_2 both in the bosonic and the supersymmetric cases. When the size of the orbifold is varied, a phase transition occurs at $L_c = 2\pi/m$, where m is the mass of ϕ. For $L < L_c$, there is a unique vacuum, while for $L > L_c$, there are two degenerate vacua. We also obtain the 1-loop quantum corrections around these vacuum solutions, exactly in the case of $L < L_c$ and perturbatively for L greater than but close to L_c. Including the fermions we find that the "chiral" zero modes around the fixed points are different for $L < L_c$ and $L > L_c$. As for the quantum corrections, the fermionic contributions cancel the singular part of the bosonic contributions at $L = 0$. Then the total quantum correction has a minimum at the critical length L_c.

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I. INTRODUCTION

The vacuum structure and the soliton solutions of a field theory can be changed dramatically in the presence of compact dimensions. A well-known example is that of the Hosotani mechanism [1] in which vanishing field strength does not necessarily imply vanishing gauge potential in nonsimply connected spaces. Then the nonvanishing gauge potential can signify the breaking of gauge symmetries. This is actually related to the fact that on nonsimply connected spaces fields can have different, or twisted, boundary conditions compatible with the gauge symmetry [2]. On the other hand, one can also study the dependence of the vacuum structure on these boundary conditions in the situations with global symmetries only [3].

The allowed soliton solutions of the theory can depend on these boundary conditions too. For example, the kink solutions of the two-dimensional ϕ^4 theory with a symmetry breaking potential will disappear when the space is compactified to a circle [4], that is, when periodic boundary condition is imposed on the scalar field. They are replaced by sphaleron solutions consisting of kink and antikink pairs. Since the space is compact, finite energy requirement no longer presents a constraint on the possible soliton solutions. Consequently, the topological classifications of these solutions have to be modified accordingly.

In addition to compactifying to a circle, one can consider that of an orbifold like S^1/Z_2. This is related to the equivalence of translations as well as reflections or parity operations in the internal dimensions. Compactifying on orbifolds was originally considered in the context of strings [5]. Recently, there are quite a lot of interests on the construction of GUT models with orbifold extra dimensions [6,7]. This is a simple way to obtain chiral zero mode fermions on the fixed points where the physical dimensions reside [8]. The Hosotani mechanism can also be realized on compact spaces like the orbifolds [9], making it possible to have symmetry breaking without the Higgs field in these models.

In order to have a more detailed understanding of the properties of the scalar as well as the fermion fields on the orbifold, we consider the simple case of the two-dimensional ϕ^4 theory in this paper. In this model most of the analysis can be carried out explicitly. On the other hand, the results that we obtain here should also be relevant to the theories in higher dimensions. In the next section, we consider the vacuum solutions and their quantum corrections on the orbifold S^1/Z_2 with only scalar fields. In Sec. III, we include fermions by introducing a supersymmetric Lagrangian. The fermionic contributions to the quantum corrections are then calculated. Conclusions and discussions are given in Sec. IV.

II. TWO-DIMENSIONAL ϕ^4 THEORY

In this section we consider the ϕ^4 theory in $(1+1)$-dimensions,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - U(\phi), \tag{1}$$

where

$$U(\phi) = \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2. \tag{2}$$

First we derive the vacuum solutions on S^1/Z_2 from the static solutions on S^1 [4] for different scales of the spatial dimension. Then the quantum corrections about these solutions are calculated using direct mode sums with zeta-function regularization [10].

A. Vacuum solutions

The equation of motion to the Lagrangian in Eq. (1) is

$$-\partial^2\phi - U'(\phi) = 0, \tag{3}$$

and if we concentrate on the static solutions, we have

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$$\frac{d^2\phi}{dx^2} = -(-U'(\phi)). \tag{4}$$

This is just the Newton's second law with x identified as "time". On the circle S^1, the solutions [4] can be readily obtained if one imposes the periodic boundary conditions, $\phi(x+L) = \phi(x)$, where L is the perimeter of the circle. They are the vacuum solutions,

$$\phi_v = \pm\frac{m}{\sqrt{\lambda}}, \tag{5}$$

the unstable solution,

$$\phi_0 = 0, \tag{6}$$

and the periodic solutions,

$$\phi_n = \frac{m}{\sqrt{\lambda}}\left(\sqrt{\frac{2k^2}{1+k^2}}\right)\mathrm{sn}\left[\frac{m}{\sqrt{1+k^2}}x\right], \tag{7}$$

where sn is the Jacobi elliptic function. ϕ_n consists of n pairs of kink and antikink. Here, the relation between L, n, and the modular parameter k ($0 \leq k \leq 1$) is

$$L = \frac{4n\sqrt{1+k^2}}{m}K(k), \tag{8}$$

where $K(k)$ is the complete elliptic integral. Since the minimum value of $K(k)$ is $K(0) = \pi/2$, the number of allowed ϕ_n solutions increases with L. For example, the one kink-anti-kink pair solution exists only when $L \geq L_1 = 2\pi/m$. On the circle, both ϕ_0 and ϕ_n are unstable and they will decay to the vacuum solutions ϕ_v. The energies of these configurations, in unit of the kink energy $E_0 = 2\sqrt{2}m^3/3\lambda$, as functions of L are plotted in Fig. 1 [4]. As shown in this figure, for $L < L_1$, ϕ_0 is the only unstable solution of the theory and it is interpreted as the sphaleron solution. While for $L > L_1$, ϕ_1 is also an unstable solution. Since it has lower energy than ϕ_0, it becomes the sphaleron solution of the theory.

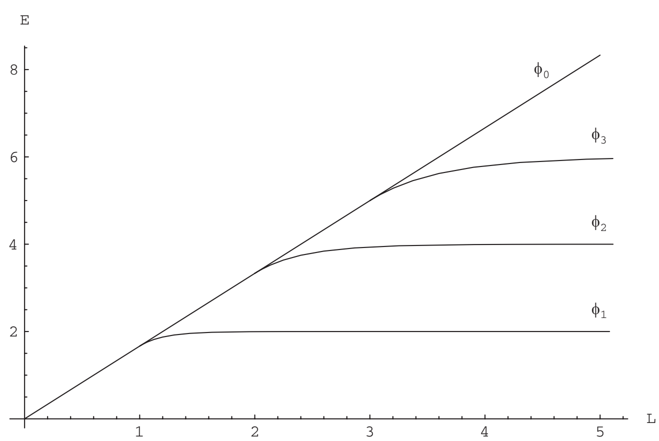

FIG. 1. Variation of the energies (in units of the kink energy $E_0 = 2\sqrt{2}m^3/3\lambda$) of the static solutions with the size (in units of the length L_1) of the circle.

On the orbifold S^1/Z_2, one identifies x and $-x$ on the basic interval $-L/2 < x < L/2$. Since the Lagrangian (Eq. (1)) is invariant under the transformation $\phi \to -\phi$, one could require

$$\phi(-x) = \pm\phi(x), \tag{9}$$

that is, one could impose either parity on the field. Not much is changed if one imposes the positive parity condition on the field. However, if one chooses instead the negative parity condition, the vacuum structure of the theory can be changed dramatically. Requiring the field to have the orbifold constraint,

$$\phi(-x) = -\phi(x), \tag{10}$$

the vacuum solutions ϕ_v are excluded because they have even parities. Then for $L < L_1$, ϕ_0 becomes the unique vacuum solution. For $L > L_1$,

$$\phi_{\pm 1} = \pm\frac{m}{\sqrt{\lambda}}\left(\sqrt{\frac{2k^2}{1+k^2}}\right)\mathrm{sn}\left[\frac{m}{\sqrt{1+k^2}}x\right], \tag{11}$$

with k implicitly given by $\sqrt{1+k^2}K(k) = mL/4$, are the degenerate vacuum solutions. Note that because of the orbifold constraint, which breaks the translational symmetry of the static solutions, ϕ_1 and ϕ_{-1} can no longer be considered as the same solution. In effect, a phase transition occurs when L is varied. The critical perimeter is

$$L_c = L_1 = \frac{2\pi}{m}. \tag{12}$$

It is interesting to note that this phase transition is specifically related to the orbifold structure of the space. It will not occur in the case of a circle with periodic boundary condition alone. Here for $L < L_c$, we have the unique vacuum ϕ_0. At L_c, ϕ_0 bifurcates into three solutions, ϕ_0, ϕ_1, and ϕ_{-1}. $\phi_{\pm 1}$ become the degenerate vacua and ϕ_0 the sphaleron solution. When the orbifold is taken as the extra dimension in a higher dimensional theory, the above phase transition will certainly be relevant to its vacuum structure. Moreover, if more fields are added to the theory, their properties will be affected by it too. We shall discuss the case of fermions in Section III. Before doing that we shall first consider the stability of these vacuum solutions and the quantum corrections to them.

The stability of the vacuum solutions can be analyzed by looking at the perturbations around them [4]. For $\phi_0 = 0$ with $L < L_1$, the perturbation equation is just

$$\frac{d^2\eta}{dx^2} + (\omega^2 + m^2)\eta = 0, \tag{13}$$

where $\phi(x,t) = \phi_0 + \eta(x)e^{-i\omega t}$. This is just a harmonic oscillator equation. Imposing the periodic boundary conditions, the solutions are simply

$$\eta_0 \sim \text{const} \tag{14}$$

with the frequency $\omega_0^2 = -m^2$, and

$$\eta_p \sim \sin\frac{2\pi p x}{L}, \qquad \cos\frac{2\pi p x}{L} \tag{15}$$

with

$$\omega_p^2 = \frac{4\pi^2 p^2}{L^2} - m^2 = m^2\left[\left(\frac{L_1}{L}\right)^2 p^2 - 1\right], p = 1, 2, \ldots \tag{16}$$

The lowest energy state is the only negative mode arising from the fact that ϕ_0 is unstable and it can decay to the vacuum solutions. Now, if the orbifold constraint is imposed, this negative mode will be excluded because it is even. ϕ_0 is therefore stable and becomes the unique ground state of the theory for $L < L_1$.

Similarly, one can analyze the stability of $\phi_{\pm 1}$ for $L > L_1$. Here the perturbation equation becomes [11]

$$\frac{d^2\eta}{dx^2} + (\omega^2 + m^2 - 3\lambda\phi_1^2)\eta = 0. \tag{17}$$

This is a Lamé equation. Its lowest five eigenfunctions in this case are given by the Lamé polynomials, and the rest by the Lamé transcendental functions [12]. The lowest energy state is a negative mode,

$$\eta_0(z) = \text{sn}^2(z) - \frac{1}{3k^2}(1 + k^2 + \sqrt{1 - k^2(1 - k^2)}), \tag{18}$$

where $z = mx/\sqrt{1 + k^2}$, with frequency

$$\omega^2 = m^2\left(1 - \frac{2\sqrt{1 - k^2(1 - k^2)}}{1 + k^2}\right) \leq 0. \tag{19}$$

This again indicates that $\phi_{\pm 1}$ are unstable on a circle and they will decay to the vacuum solutions. Note that this negative mode has even parity. The orbifold constraint will also exclude this mode and render $\phi_{\pm 1}$ stable. The next state is the zero mode with

$$\eta_1(z) = \text{cn}(z)\text{dn}(z). \tag{20}$$

This state is also even. Its presence is related to the translational (or rotational) symmetry of ϕ_n. The orbifold constraint will exclude this mode too because the orbifold constraint also breaks the translational symmetry of ϕ_n.

The other three Lamé polynomial states are

$$\eta_3(z) = \text{sn}(z)\text{dn}(z)\ ; \quad \omega_3^2 = \frac{3m^2k^2}{1 + k^2} \tag{21}$$

$$\eta_4(z) = \text{sn}(z)\text{cn}(z)\ ; \quad \omega_4^2 = \frac{3m^2}{1 + k^2} \tag{22}$$

which are both odd, and

$$\eta_5(z) = \text{sn}^2(z) - \frac{1}{3k^2}(1 + k^2 - \sqrt{1 - k^2(1 - k^2)}) \tag{23}$$

which is even, with

$$\omega_5^2 = m^2\left(1 + \frac{2\sqrt{1 - k^2(1 - k^2)}}{1 + k^2}\right). \tag{24}$$

The orbifold constraint will also exclude $\eta_5(z)$. Although the rest of the spectrum is not known explicitly, one can see that the parities of the eigenfunctions have the pattern: even, even, odd, odd, even, even, ... Hence, exactly half of the spectrum will satisfy the orbifold constraint.

B. Quantum corrections

Next we calculate the one-loop quantum corrections to the vacuum energies. For $L < L_1$, the vacuum solution is ϕ_0 with the classical energy

$$M_{cl}[\phi_0] = \frac{m^4 L}{4\lambda}. \tag{25}$$

The quantum correction to this energy can be evaluated explicitly using the spectrum of perturbations in Eq. (16),

$$\begin{aligned}(\Delta M)_{\phi_0} &= \frac{mL_1}{2L}\sum_{p=1}^{\infty}\left[p^2 - \left(\frac{L}{L_1}\right)^2\right]^{1/2} \\ &= \lim_{s\to -1}\frac{1}{2}\left(\frac{mL_1}{L}\right)^{-s}\sum_{p=1}^{\infty}\left[p^2 - \left(\frac{L}{L_1}\right)^2\right]^{-s/2}. \end{aligned}\tag{26}$$

Here we have used the zeta-function regularization method [10]. In terms of the Riemann zeta-function, the above sum can be expressed as

$$\begin{aligned}(\Delta M)_{\phi_0} = \lim_{s\to -1}\Bigg[&\frac{1}{2}\left(\frac{mL_1}{L}\right)^{-s}\zeta(s) + \left(\frac{L}{2L_1}\right)^2 s\left(\frac{mL_1}{L}\right)^{-s} \\ &\times \zeta(2 + s) + \left(\frac{mL_1}{L}\right)^{-s} \\ &\times \sum_{n=2}^{\infty}\frac{(-1)^n\Gamma(1 - s/2)}{2\Gamma(n + 1)\Gamma(1 - n - s/2)}\left(\frac{L}{L_1}\right)^{2n} \\ &\times \zeta(2n + s)\Bigg]. \end{aligned}\tag{27}$$

The first term,

$$\frac{1}{2}\left(\frac{mL_1}{L}\right)\zeta(-1) = -\frac{\pi}{12L}, \tag{28}$$

is rendered finite by the zeta-function regularization used here. No renormalization is needed to deal with the usual quadratic divergence for this term [13]. Note that this finite result is just the Casimir energy of a massless scalar field on the orbifold and it diverges as $L \to 0$. The second term is

$$\begin{aligned}&\lim_{s\to -1}\left(\frac{L}{2L_1}\right)^2 s\left(\frac{mL_1}{L}\right)^{-s}\zeta(2 + s) \\ &= \lim_{s\to -1}\frac{m^2 L}{8\pi}\left[-\frac{1}{s + 1} + 1 - \gamma + \ln\left(\frac{2\pi}{L}\right)\right]. \end{aligned}\tag{29}$$

This term with the pole divergent part can be cancelled by appropriately choosing the mass renormalization scheme for the two-dimensional ϕ^4 theory. The last term gives

$$m\sum_{n=2}^{\infty}\frac{(-1)^n\sqrt{\pi}}{4\Gamma(n+1)\Gamma(\frac{3}{2}-n)}\left(\frac{L}{L_1}\right)^{2n-1}\zeta(2n-1) \equiv mf(L/L_1), \tag{30}$$

where, for $0 \le L \le L_1$, $f(L/L_1)$ is a convergent series, which is plotted in Fig. 2, with $f(1) = -0.264$. Putting all these together, after mass renormalization,

$$(\Delta M)^{\rm ren}_{\phi_0} = -\frac{\pi}{12L} + mf(L/L_1), \tag{31}$$

which is plotted in Fig. 3. Here we have

$$\lim_{L\to 0}(\Delta M)^{\rm ren}_{\phi_0} = -m\left(\frac{L_1}{24L}\right); \quad \lim_{L\to L_1}(\Delta M)^{\rm ren}_{\phi_0} = -0.306m. \tag{32}$$

For $L > L_1$, the vacuum solutions are $\phi_{\pm 1}$ with their classical energies shown in Fig. 1. Since the spectrum of perturbations in this case is not known analytically, we cannot compute the quantum corrections to this energy explicitly as we have done above for ϕ_0. One way to estimate the behavior of the quantum corrections to the energy of $\phi_{\pm 1}$ is by calculating them as series in powers of k, the modular parameter of the elliptic functions. As shown in [4], one can evaluate the eigenvalues of the perturbations of $\phi_{\pm 1}$ in powers of k. To order k^2, we have,

$$\omega_p^2 = \begin{cases} 3k^2m^2, & p = 1 \\ m^2(p^2-1) - \frac{3}{2}k^2m^2(p^2-2), & p = 2, 3, \ldots \end{cases} \tag{33}$$

Hence,

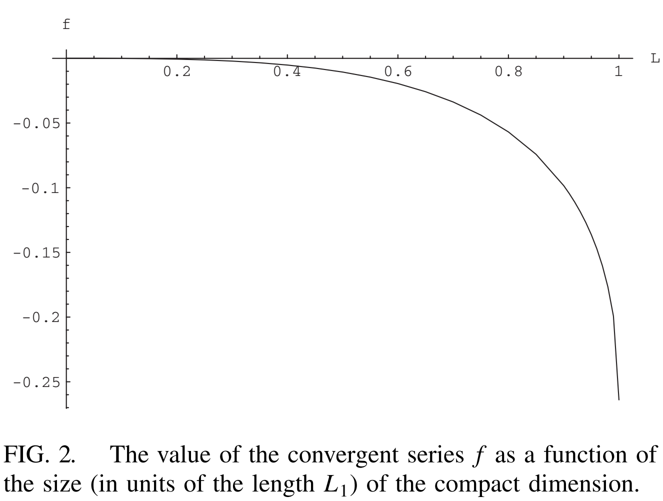

FIG. 2. The value of the convergent series f as a function of the size (in units of the length L_1) of the compact dimension.

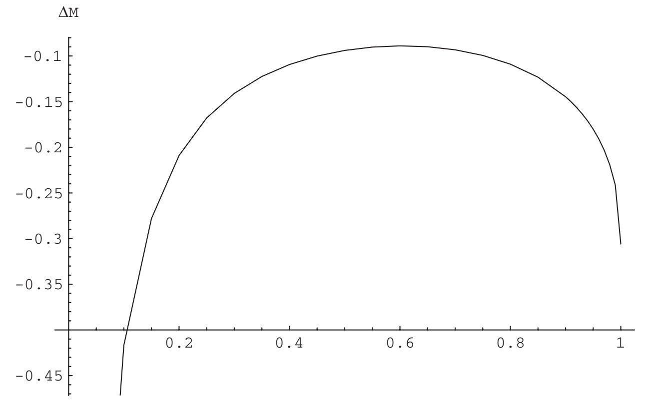

FIG. 3. The value of the 1-loop quantum corrections (in unit of the mass parameter m) as a function of the size (in units of the length L_1) of the compact dimension.

$$(\Delta M)^{\rm ren}_{\phi_{\pm 1}} = (\Delta M)^{\rm ren}_{\phi_0}\Big|_{L=L_1} + m\left(\frac{\sqrt{3}}{2}k\right) + O(k^2) = m\left(-0.306 + \frac{\sqrt{3}}{2}k + O(k^2)\right) \tag{34}$$

where only the term with $p = 1$ contributes to $(\Delta M)^{\rm ren}_{\phi_{\pm 1}}$ to order k. From Fig. 3, we can see that for $L < L_1$ the quantum corrections to the vacuum energy decreases with L as L approaches L_1. On the other hand, from Eq. (34), for $L > L_1$, the quantum corrections increases with k or L as L increases from L_1. Thus, there is a discontinuity in the first derivative of the total vacuum energies at $L = 2\pi/m$ (Eq. (12)), indicating again the presence of a phase transition that we have mentioned before.

III. INCLUDING FERMIONS

To include fermions in our model, we consider the (1+1)-dimensional supersymmetric Lagrangian [14,15],

$$L = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}W^2 + \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{1}{2}W'\bar{\psi}\psi, \tag{35}$$

where the superpotential

$$W = \sqrt{\frac{\lambda}{2}}\left(\phi^2 - \frac{m^2}{\lambda}\right) \tag{36}$$

Here ψ is a Majorana spinor. Note that the bosonic part is the same as the model in the last section. This Lagrangian is invariant under the supersymmetric transformation,

$$\delta\phi = \bar{\epsilon}\psi\delta\psi = -(i\gamma^\mu\partial_\mu\phi + W)\epsilon, \tag{37}$$

with the corresponding supersymmetric current,

$$J^\mu = (\gamma^\nu \partial_\nu \phi + iW)\gamma^\mu \psi, \tag{38}$$

and the supercharges,

$$Q = \int dx J^0 = \int dx(\gamma^\nu \partial_\nu \phi + iW)\gamma^0 \psi. \tag{39}$$

The supersymmetric algebra reads

$$Q_1^2 = Q_2^2 = 2H, \{Q_1, Q_2\} = 0 \tag{40}$$

where H is the Hamiltonian,

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

and we have chosen the gamma matrices $\gamma^0 = \sigma_2$ and $\gamma^1 = i\sigma_3$.

On the orbifold S^1/Z_2, we can see that under the transformation,

$$\phi(-x) = -\phi(x), \psi(-x) = \pm \sigma_3 \psi(x) \tag{41}$$

the supersymmetric Lagrangian in Eq. (35) is invariant. Hence, we can choose the orbifold constraint for the fermionic field as

$$\psi(-x) = \sigma_3 \psi(x) \Rightarrow \begin{pmatrix} \psi_1(-x) \\ \psi_2(-x) \end{pmatrix} = \begin{pmatrix} \psi_1(x) \\ -\psi_2(x) \end{pmatrix}, \tag{42}$$

that is, ψ_1 is even and ψ_2 is odd. We could have chosen a minus sign in Eq. (42) for the fermionic field. This would only interchange the roles of ψ_1 and ψ_2.

As in the bosonic case, the vacuum solution for $L < L_1$ is ϕ_0, while for $L > L_1$, they are $\phi_{\pm 1}$. The energies of these solutions are all nonzero so supersymmetry is broken by these vacua [15].

To calculate the quantum corrections, we consider first ϕ_0 for $0 \geq L \geq L_1$. The bosonic spectrum is again given by Eq. (16). For the fermionic perturbations $u(x, t) = u(x)e^{-i\omega_F t}$, we have the equation of a massless fermion,

$$\begin{aligned} \frac{du_1}{dx} &= -i\omega_F u_2, \qquad -\frac{du_2}{dx} = i\omega_F u_1 \\ \Rightarrow -\frac{d^2u_1}{dx^2} &= \omega_F^2 u_1, \qquad -\frac{d^2u_2}{dx^2} = \omega_F^2 u_2. \end{aligned} \tag{43}$$

Because of the orbifold constraint, u_1 must be even and u_2 must be odd. Therefore, only one component of the fermionic perturbation, u_1, can develop a zero mode,

$$u_1 \sim \text{const} \tag{44}$$

while u_2 cannot. This is the same mechanism to obtain chiral fermion on the fixed points of the orbifold in the $(1 + 4)$-dimensional setting [8]. Here in $(1 + 1)$-dimensions, we have

$$\gamma^1 u_1 = u_1, \qquad \gamma^1 u_2 = -u_2. \tag{45}$$

For the positive modes, we have

$$u_1 \sim \cos\frac{2\pi p}{L}x; \qquad u_2 \sim \sin\frac{2\pi p}{L}x \tag{46}$$

and eigenvalues

$$\omega_F^2 = \frac{4\pi^2 p^2}{L^2}, \qquad p = 1, 2, \ldots \tag{47}$$

The quantum correction is thus

$$\begin{aligned} (\Delta M)_{\phi_0}^{SUSY,\text{ren}} &= \frac{1}{2}\sum \omega_B - \frac{1}{2}\sum \omega_F \\ &= \frac{mL_1}{2L}\sum_{p=1}^{\infty}\left\{\left[p^2 - \left(\frac{L}{L_1}\right)^2\right]^{1/2} - p\right\} \end{aligned} \tag{48}$$

Using the same mass renormalization procedure as in the bosonic case in the last section, we see that the fermionic contribution just cancels the Casimir energy term. Hence, we have

$$(\Delta M)_{\phi_0}^{SUSY,\text{ren}} = mf(L/L_1), \tag{49}$$

where $f(L/L_1)$ is the function in Fig. 2. The divergence at $L = 0$ is thus cured by the inclusion of fermions in the supersymmetric Lagrangian.

Next we consider the quantum corrections to the vacuum solutions $\phi_{\pm 1}$ for $L > L_1$. As discussed in the last section, the equation of the bosonic perturbations is the Lamé equation. For the fermionic perturbations, we have

$$\begin{aligned} \left(\frac{d}{dx} + \sqrt{2\lambda}\phi_1\right)u_1 &= -i\omega_F u_2, \\ \left(-\frac{d}{dx} + \sqrt{2\lambda}\phi_1\right)u_2 &= i\omega_F u_1 \end{aligned} \tag{50}$$

Although these equations are not exactly solvable, the zero modes can nevertheless be obtained simply as [16]

$$u_{10} \sim e^{-\int^x \sqrt{2\lambda}\phi_1}, \qquad u_{20} \sim e^{\int^x \sqrt{2\lambda}\phi_1} \tag{51}$$

Both of these zero modes are even. If the orbifold constraint is imposed, only u_{10} survives. The situation is thus the same as that for ϕ_0 with $L < L_1$.

From Eq. (50), we see that the eigenstates of u_1 and u_2 are related by

$$u_{2n} \sim \left(\frac{d}{dx} + \sqrt{2\lambda}\phi_1\right)u_{1n} \tag{52}$$

Hence, u_{2n} will be automatically parity odd if u_{1n} is parity even.

As in the bosonic case, we can consider the quantum corrections of the fermions to the vacuum energy perturbatively when k is small. Here we see that the zero modes remain to have zero energies for all values of k (Eq. (51)). For the higher modes, the corrections are at least of order k^2 by direct perturbative calculations similar to the bosonic case. Hence, the quantum corrections to ω_F are at least of the order k^2, while the bosonic ones are of the order k as shown in Eq. (34). To the lowest order of k, we finally have

$$(\Delta M)^{SUSY,\mathrm{ren}}_{\phi_{\pm 1}} = m\left(f(1) + \frac{\sqrt{3}}{2}k + O(k^2)\right) = m\left(-0.264 + \frac{\sqrt{3}}{2}k + O(k^2)\right) \quad (53)$$

IV. CONCLUSIONS AND DISCUSSIONS

We have considered the vacuum structure of the (1+1)-dimensional ϕ^4 theory on the orbifold S^1/Z_2. When the size of the orbifold is varied, we have found that a phase transition occurs at $L = L_c = 2\pi/m$. For $L < L_c$, there is a unique classical vacuum solution $\phi_0 = 0$, while for $L > L_c$, there are two degenerate vacua, ϕ_1 and ϕ_{-1}. It is worth to note that this phase transition occurs only after imposing the orbifold constraint together with the periodic boundary condition. This phenomenon has been overlooked before because L is usually taken as finite but large in constructing orbifold field theory models [8]. We think that this phase transition will be important when one considers the case of dynamical compact dimensions, especially in the cosmological setting in the early universe.

We have also calculated the quantum corrections to the vacuum solutions from the bosonic contributions using zeta-function regularization to deal with the divergent quantities. As shown in Fig. 3, the correction is dominated by the Casimir energy for small L and it goes to negative infinity as $L \to 0$. On the other hand, as $L \to L_c$, the correction decreases to a finite value. For $L > L_c$, we use perturbative method to estimate the quantum correction. For L close to L_c, the correction increases with L. Hence, the quantum correction has a dip at $L = L_c$ with a discontinuity in the slope, which is another indication of the presence of the phase transition.

Fermions are included in the model by using a supersymmetric Lagrangian. Since the vacuum solutions all have nonzero energies, the supersymmetries, with supercharges Q_1 and Q_2, are broken. With the fermions the main difference is that the Casimir energy in the quantum corrections is cancelled by the fermionic contributions. Then the correction goes to zero, instead of negative infinity, as $L \to 0$.

The results obtained here should also be relevant to cases in higher dimensions. For example, in the case of five dimensions with one space compactified to an orbifold, the fields there can be expressed as products of a four-dimensional part and another part which depends only on the orbifold dimension [8]. Then one needs to consider the two different vacuum structures for the size of the internal dimension smaller or larger than L_c. Moreover, the fermionic zero modes (Eqs. (44) and (51)) are different in these two cases. In fact, they have different forms around the fixed points at $x = 0$ and $x = L/2$ which means that they could have different phenomenology on the physical dimensions.

Other than going to higher dimensions, it is interesting to see what the vacuum structures as well as the soliton solutions of the theory when gauge fields are included on the orbifold. In [17], the monopole string solution, which is independent of the compact dimension, is generalized to the orbifold case. In that respect, one can also look at the instanton, or caloron, solutions with or without nontrivial holonomy, that is, whether there is symmetry breaking or not [18,19]. We hope to look at these cases in the future publications.

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[1] Y. Hosotani, Phys. Lett. B **126**, 309 (1983); Ann. Phys. (N.Y.) **190**, 233 (1989).
[2] J. Scherk and J. H. Schwarz, Phys. Lett. B **82**, 60 (1979).
[3] H. Hatanaka, S. Matsumoto, K. Ohnishi, and M. Sakamoto, Phys. Rev. D **63**, 105003 (2001).
[4] N. S. Manton and T. M. Samols, Phys. Lett. B **207**, 179 (1988).
[5] L. J. Dixon, J. A. Harvey, C. Vafa, and E. Witten, Nucl. Phys. **B261**, 678 (1985); **B274**, 285 (1986).
[6] Y. Kawamura, Prog. Theor. Phys. **105**, 999 (2001).
[7] M. Quiros, hep-ph/0302189.
[8] H. Georgi, A. K. Grant, and G. Hailu, Phys. Rev. D **63**, 064027 (2001); Phys. Lett. B **506**, 207 (2001).
[9] N. Haba, M. Harada, Y. Hosotani, and Y. Kawamura, Nucl. Phys. **B657**, 169 (2003); **B669**, 381(E) (2003).
[10] See, e.g., E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, and S. Zerbini, *Zeta Regularization Techniques with Applications* (World Scientific, Singapore, 1994).
[11] J.-Q. Liang, H. J. W. Müller, and D. H. Tchrakian, Phys. Lett. B **282**, 105 (1992).

[12] F. M. Arscott, *Periodic Differential Equations* (Pergamon, Oxford, 1964).
[13] R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D **10**, 4130 (1974).
[14] P. Di Vecchia and S. Ferrara, Nucl. Phys. **B130**, 93 (1977).
[15] E. Witten and D. I. Olive, Phys. Lett. B **78**, 97 (1978).
[16] F. Cooper, A. Khare, and U. Sukhatme, *Supersymmetry and Quantum Mechanics* (North-Holland, Amsterdam, 1995).
[17] R. Dermisek, S. Raby, and S. Nandi, Nucl. Phys. **B641**, 327 (2002).
[18] D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. **53**, 43 (1981).
[19] T. C. Kraan and P. van Baal, Nucl. Phys. **B533**, 627 (1998).